# Understanding Options and Their Role in Hedging via the "Greeks"

Bradley J. Wogsland
*Department of Physics and Astronomy,*
*University of Tennessee, Knoxville, TN 37996-1200*

Options are priced assuming that the underlying stock price obeys Brownian motion with a normal Gaussian distribution plus drift and that arbitrage opportunities do not exist. The Black-Scholes method of option pricing is elucidated through the five "Greeks": delta, theta, gamma, vega, and rho. Knowledge of these can be used to hedge against risk.

## I. INTRODUCTION

When in 1973 Fisher Black and Myron Scholes published their option pricing formula [1] the world was taken by storm. Using stochastic calculus, they were able to derive the formula for a European call option by assuming that the possibility of arbitrage does not exist. Arbitrage is defined as the opportunity for a risk-free profit, that is, a profit over and above the risk-free interest rate. The derivation here is a variation of that found in [2]. In this model, the stock price is given by

$$dS = \mu S dt + \sigma S dz \tag{1}$$

where S is the price of the stock, z is the Brownian motion, $\mu$ is the expected rate of return and $\sigma$ is the volatility. Using Ito's Lemma [3], this yields the price of a derivative f (S,t) to be

$$df = ((\partial f/\partial S)\mu S + \partial f/\partial t + .5(\partial^2 f/\partial S^2)\sigma^2 S^2)dt + (\partial f/\partial S)\sigma S dz. \tag{2}$$

We want to eliminate the Brownian motion from these two equations and we can do so by creating a portfolio P(f, S, t) of 1 derivative and $\partial f/\partial S$ shares of stock. Now this gives the instantaneous value of the portfolio to be

$$dP = df - (\partial f/\partial S)dS$$

$$= (\partial f/\partial t + .5(\partial^2 f/\partial S^2)\sigma^2 S^2)dt. \tag{3}$$

Here is where the no arbitrage assumption is applied. Under this assumption the return on P can be no greater than the riskless rate of return r. Thus,

$$dP = rPdt \tag{4}$$

$$(\partial f/\partial t + .5(\partial^2 f/\partial S^2)\sigma^2 S^2)dt = r\,(f - (\partial f/\partial S)S)dt$$

$$rf = \partial f/\partial t + (\partial f/\partial S)S + .5(\partial^2 f/\partial S^2)\sigma^2 S^2. \tag{5}$$

This is commonly referred to as the Black-Scoles-Merton equation [4], which all derivatives must satisfy in addition to their own specific boundary conditions.

The technique used by Black and Scholes in this general form applies to a cornucopia of other financial derivatives, however, the subject of the original Black-Scholes paper [1] was European call options. The price V of the European call option they derived was

$$V = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-) \tag{6}$$

where $S_0$ is the initial price of the stock, K is the strike price, $\Phi(x)$ is the cumulative distribution function and

$$d_\pm = (\ln(S_0/K) + (r \pm \sigma^2/2)/T)/(\sigma\sqrt{T}) \tag{7}$$

where $\sigma$ is the volatility, and T is the time till expiry.

## II. DELTA

Various partial derivatives of the option price V are used to understand and estimate its future behavior. Denoted by letters from the Greek alphabet, they are affectionately called the "Greeks" in financial circles. The first of these to be discussed here is $\Delta$, which is defined by

$$\Delta = \partial V/\partial S \tag{8}$$

where S is the price of the underlying stock.

Delta is primary used for the aptly named delta hedging. This is actually what was done above in deriving (5), setting the number of shares to be held in the portfolio to hedge one derivative to be $-\Delta$. Obviously, $\Delta$ changes as the option price V changes. The derivation of the Black-Scholes-Merton equation assumes that a trader can delta hedge instantaneously as $\Delta$ changes. In practice, delta hedging cannot be done instantaneously, as individual trades must involve non-infinitesimal share quantities and finite time separations. A hedge and forget strategy involves only one trade at time $t = 0$ and minimizes transaction costs, which can be prohibitive for smaller investments. For larger investments a dynamic hedging strategy, involving several trades at various intervals, is more appropriate. This begs the question of how large the interval between trades ought to be. To estimate this we need gamma.

## III. GAMMA

Next of the "Greeks" to come within our ken is $\Gamma$, defined by

$$\Gamma = \partial^2 V/\partial S^2 \tag{9}$$

where S is again the price of the underlying stock. Note that gamma is also the rate of change in delta with stock price holding all else constant,

$$\Gamma = \partial\Delta/\partial S. \tag{10}$$

Furthermore, we can now rewrite (5) in terms of $\Delta$ and $\Gamma$,

$$rV = \partial V/\partial t + \Delta S + .5\Gamma\sigma^2 S^2. \qquad (5')$$

The real utility of gamma comes from the Taylor expansion of the portfolio P(S) given by

$$dP = (\partial P/\partial S)dS + .5(\partial^2 P/\partial S^2)dS^2 + \ldots \qquad (11)$$

Having a portfolio delta neutral eliminates the first term. To eliminate the second term we note that

$$\Gamma_V = \partial^2 V/\partial S^2 \neq \partial^2 P/\partial S^2 \qquad (12)$$

which means that $-\Gamma/\Gamma_V$ options added to the portfolio will do the trick. However, because the Brownian motion term dz in (1) is proportional to $\sqrt{dt}$, $dS^2$ has a term of order dt. Thus we really ought to consider time better in our analysis. This leads us to our next "Greek".

## IV. THETA

The third "Greek" to be discussed here is $\Theta$, which is often called the time decay of an option in a rather pessimistic fashion. $\Theta$ is defined by

$$\Theta = \partial V/\partial T \qquad (13)$$

where T is the time until the option expires. Now we can write (5) in it's final form:

$$rf = \Theta + \Delta S + .5\Gamma\sigma^2 S^2 \qquad (5'')$$

for a general derivative f(S, T). In particular, for a delta-hedged portfolio,

$$rP = \Theta + .5\Gamma\sigma^2 S^2 \qquad (14)$$

which shows that if all else is constant a change in $\Theta$ in one direction will mean a change in $\Gamma$ in the opposite direction. This is useful on the trading floor because $\Theta$ is easier to measure than $\Gamma$.

Now modifying our Taylor expansion (11) for the portfolio P(S, T),

$$dP = (\partial P/\partial S)dS + (\partial P/\partial T)dT + .5(\partial^2 P/\partial S^2)dS^2 + \ldots \qquad (15)$$

This shows that having a delta and gamma hedged portfolio is not enough to perfectly hedge the portfolio to second order in S, and no strategy has yet been produced to theta hedge.

## V. VEGA

The next variable we wish to consider is one that we have heretofore claimed, especially in our derivation of the Black-Scholes formula, to be constant. This is the volatility σ. Continuing our Taylor expansion strategy for the portfolio P(S, σ, T),

$$dP = (\partial P/\partial S)dS + (\partial P/\partial \sigma)d\sigma + (\partial P/\partial T)dT + .5(\partial^2 P/\partial S^2)dS^2 + \ldots \qquad (16)$$

This allows us to introduce the penultimate "Greek" letter which comes up, which is ironically not actually a Greek letter at all. Vega, commonly denoted by κ, is defined to be

$$\kappa = \partial V/\partial \sigma \qquad (17)$$

where σ is volatility of the underlying stock price. Vega will bring us one step closer to the unattainable perfect hedge. Vega hedging also adds another level of complexity to our portfolio, however, as an option on a second stock must be purchased. The amount to be purchased is $-\kappa_P/\kappa_2$ where $\kappa_P$ is the vega of P and $\kappa_2$ is the vega of the second option. A vega hedge can also be done with a single option, but in general

$$-\kappa_P/\kappa_2 \neq -\Gamma/\Gamma_V \qquad (18)$$

so in this case there can either be a vega or a gamma hedge.

## VI. RHO

Last but not least among the "Greeks" is ρ, defined by

$$\rho = \partial V/\partial r \qquad (19)$$

where r is the riskless interest rate. The careful reader will also note that this quantity was also required to be constant in the derivation of the Black-Scholes formula. We can now write the Taylor expansion of P(r, σ, S, T) in its final form including all the "Greeks":

$$dP = \Delta dS + \rho dr + \kappa d\sigma + \Theta dT + .5\Gamma dS^2 + \ldots \qquad (20)$$

Similarly to theta, rho cannot in practice be hedged.

## VII. HEDGING IN PRACTICE: AN EXAMPLE

For definiteness in this example we shall concern ourselves with the European call option, the price of which is given by (6). First we consider the most important hedge: the delta hedge. Applying (8) to this option price formula,

$$\Delta = (\partial/\partial S)(S\Phi(d_+) - Ke^{-rT}\Phi(d_-))$$

$$\Delta = \Phi(d_+). \qquad (21)$$

So, for every call option bought, $\Delta$ shares of the underlying stock must be shorted (or, for every call option written, $\Delta$ shares of the underlying stock must be bought) to maintain a delta neutral portfolio. As a numerical example, consider a purchase of 100 options on a stock with price S = 40, volatility $\sigma = 3$, strike price K = 44, and time till expiry T = 10 where the riskless interest rate is r = .05. Applying (21) delta is found to be $\Delta \approx .52$, so . 52x100 = 52 shares of stock must be shorted to hedge to purchase of the options. In practice large portfolio managers can afford the transaction costs of doing this daily.

The next order correction is the gamma hedge. Applying (10) to the above,

$$\Gamma_V = (\partial/\partial S)\Phi(d_+)$$

$$\Gamma_V = \Phi'(d_+)/S_0\sigma\sqrt{T}. \tag{22}$$

Basically, this means that for every change in the stock price $\delta S$, $\Delta$ increases by $\Gamma_V\delta S$. In a simple portfolio with just positions in a call option and its underlying stock it doesn't make sense to talk about gamma neutrality, but As a numerical example, consider a diverse portfolio with overall $\Gamma = -100$. Using a call option on the same stock as above, we find that $\Gamma_V \approx .001$ for this option. Thus to gamma hedge the portfolio, we to purchase 100/.001 = 100,000 call options. Furthermore, to delta hedge this transaction, about 52,000 shares of the underlying stock must be shorted. Obviously, it would be better to choose another option to make our portfolio neutral! In practice, the transaction costs of gamma hedging are prohibitive even for large portfolios, so gamma is usually watched, hedging only when it deviates to far from zero.

As mentioned earlier, $\Theta$ and $\rho$ cannot be hedged against, only monitored. Basically the only options available if they go to far afield is to jump ship or ride out the storm. For the European call option, applying (13)

$$\Theta = (\partial/\partial T)(S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-))$$

$$\Theta = -S_0\sigma\Phi'(d_+)/2\sqrt{T} - rKe^{-rT}\Phi(d_-) \tag{23}$$

which, in the case of our example stock, becomes $\Theta \approx -8.26$. Similarly, applying (19) to the call option price formula

$$\rho = (\partial/\partial r)(S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-))$$

$$\rho = TKe^{-rT}\Phi(d_-) \tag{24}$$

which, again in the case of our example stock, becomes $\rho \approx 139$.

Lastly, the vega hedge, like the gamma hedge, involves modifying a position in a call option. Using (17),

$$\kappa = (\partial/\partial\sigma)(S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-))$$

$$\kappa = -S_0\Phi'(d_+)\sqrt{T}. \tag{25}$$

As with gamma, for a simple two item portfolio vega is meaningless. Furthermore, (18) shows that an option will in general not be able to be used to produce both a gamma and a vega hedge. Suppose we have a large portfolio with $\kappa = 2700$ and we want to hedge using the above mentioned European call option. The vega of this option is $\kappa_V \approx -50$, yielding that -2700/-50 = 54 options must be purchased to vega hedge this portfolio. Also 28 shares of stock must be shorted to preserve delta-neutrality. Clearly, this option is a good choice for a vega hedge.

The entire discussion of this section transfers rather straightforwardly to other non-linear derivatives in an underlying stock. Equations (21) – (25) will, of course, take on a somewhat different form based on the derivative used, but are in most cases no more difficult to derive from the derivative's price formula than the above.

## VIII. CONCLUSIONS

Like any good scientific theory the Black-Scholes formula has been born out by experiment, however there are significant flaws in some of the assumptions involved. Mandelbrot [5] has pointed out that stock prices most likely do not follow a Brownian-motion-like path, that is, the volatility is not constant but varies over time. This is perhaps why Black-Scholes analysis failed to predict an event like Black Friday in 1987, or the failure of Long Term Capital Management in 1998. Nevertheless, their options pricing formula represents a good basic theory to use most of the time, and doubtlessly proper refinements will bring current outliers back within the fold.

The "Greeks" provide the best known way to analyze derivative pricing formulas such as Black-Scholes. They can be evidenced by Taylor expanding the price. The main utility of the "Greeks" involves hedging. In practice most large portfolios delta hedge on a daily basis and monitor vega and gamma to see that they don't get too big, hedging when necessary. Generally, smaller traders cannot afford the transaction costs of any strategy other than hedge and forget. Rho and theta cannot be hedged, only monitored.

---


[1] F. Black and M. Scholes, Journal of Political Economy **81**, 637 (1973).

[2] J. C. Hull, *Options, Futures, and Other Derivatives* 5th ed., (2002).

[3] K. Ito, Memoirs, American Mathematical Society **4**, 1 (1951).

[4] R. C. Merton, Bell Journal of Economics and Management Science **4**, 141 (1973).

[5] B. Mandelbrot and R. L. Hudson, *The (Mis)Behavior of Markets*, (2004).